

8.1 - Preliminary Theory-Linear Systems

Parallels exist between concepts, theorems, and vocabulary of linear differential equations and linear systems of differential equations:

- Existence and uniqueness of solutions
- The superposition principle
- Linear dependence/independence and how they relate to the Wronskian (though the Wronskian is different here)
- Fundamental set of solutions
- Homogeneous and nonhomogeneous systems
- Complementary and particular solutions

Recall from algebra: A system of equations can be represented as a matrix equation. For instance, the system

$$\begin{cases} 2x + 3y - 5z = 12 \\ 7x + 4y + 6z = 18 \\ -2x + z = -27 \end{cases} \text{ can be represented by the matrix equation}$$

$$\begin{pmatrix} 2 & 3 & -5 \\ 7 & 4 & 6 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix} \rightarrow \begin{pmatrix} 2x+3y-5z \\ 7x+4y+6z \\ -2x+z \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ -27 \end{pmatrix}$$

and more generally as $A\vec{X} = \vec{B}$ ← constant matrix

$\begin{matrix} \nearrow & \nwarrow \\ \text{coefficient matrix} & \text{variable matrix} \end{matrix}$

In this chapter, we will primarily be concerned with linear systems of DEs that have the forms

$$X' = AX \quad \text{and} \quad X' = AX + F$$

$$\vec{X}' = A\vec{X}$$

Homogeneous system

$$\vec{X}' = A\vec{X} + \vec{F}(t)$$

Nonhomogeneous system

$$\frac{dx}{dt} = 3x + 5y + t$$

$$\frac{dy}{dt} = 2x - 7y + 4t$$

Example: Write the given linear system in matrix form.

$$\begin{cases} \frac{dx}{dt} = 4x - 7y \\ \frac{dy}{dt} = 5x \end{cases}$$

Form: $\vec{X}' = A \vec{X}$

$$\vec{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \vec{X}$$

Example: Write the given linear system in matrix form.

$$\begin{cases} \frac{dx}{dt} = -3x + 4y + e^{-t} \sin 2t \\ \frac{dy}{dt} = 5x + 9z + 4e^{-t} \cos 2t \\ \frac{dz}{dt} = y + 6z - e^{-t} \end{cases}$$

Non homogeneous

$$\vec{X}' = A \vec{X} + \vec{F}(t)$$

$$\vec{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \vec{X} + \begin{pmatrix} e^{-t} \sin 2t \\ 4e^{-t} \cos 2t \\ -e^{-t} \end{pmatrix}$$

OR
$$\vec{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \vec{X} + e^{-t} \begin{pmatrix} \sin 2t \\ 4 \cos 2t \\ -1 \end{pmatrix}$$

Example: Write the given linear system without the use of matrices.

$$\mathbf{X}' = \begin{pmatrix} 7 & 5 & -9 \\ 4 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{5t} - \begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix} e^{-2t}$$

$$\frac{dx}{dt} = 7x + 5y - 9z - 8e^{-2t}$$

$$\frac{dy}{dt} = 4x + y + z + 2e^{5t}$$

$$\frac{dz}{dt} = -2y + 3z + e^{5t} - 3e^{-2t}$$

Example: Verify that the vector $\mathbf{X}$ is a solution of the given homogeneous linear system.

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X}; \quad \mathbf{X} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^t$$

$$\vec{X} = \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix}$$

$$\vec{X}' = \begin{pmatrix} e^t + 4e^t + 4te^t \\ 3e^t - 4e^t - 4te^t \end{pmatrix} = \begin{pmatrix} 5e^t + 4te^t \\ -e^t - 4te^t \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \vec{X} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^t + 4te^t \\ 3e^t - 4te^t \end{pmatrix}$$

$$= \begin{pmatrix} 2e^t + 8te^t + 3e^t - 4te^t \\ -e^t - 4te^t \end{pmatrix} = \begin{pmatrix} 5e^t + 4te^t \\ -e^t - 4te^t \end{pmatrix}$$

Match ✓